

ON FINITELY GENERATED MODELS OF THEORIES WITH AT MOST COUNTABLY MANY NONISOMORPHIC FINITELY GENERATED MODELS

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ABSTRACT. We study finitely generated models of countable theories, having at most countably many nonisomorphic finitely generated models. We introduce a notion of rank of finitely generated models and we prove, when T has at most countably many nonisomorphic finitely generated models, that every finitely generated model has an ordinal rank. This rank is used to give a property of finitely generated models analogue to the Hopf property of groups and also to give a necessary and sufficient condition for a finitely generated model to be prime of its complete theory. We investigate some properties of limit groups of equationally noetherian groups, in respect to their ranks.

1. INTRODUCTION

Throughout this paper, we let L be a fixed countable first order language; L is arbitrary but is held fixed to simplify notation. We shall say that a model \mathcal{M} is *n-generated* if it is generated by n elements; that is, if there exists an n -tuple \bar{a} in \mathcal{M} such that for every $y \in \mathcal{M}$, $y = \tau(\bar{a})$ for some term $\tau(\bar{x})$ of L . It is worth mentioning that this definition is different from the one used by A. Pillay in [Pil81, Pil83]. A theory T is said *n-consistent* if it has an n -generated model.

The purpose of the paper is to study some properties of finitely generated models of countable theories which have at most countably many nonisomorphic finitely generated models. Part of our interest on this case comes from the fact that the number of nonisomorphic finitely generated models of a countable theory is either at most \aleph_0 or equals to 2^{\aleph_0} (Theorem 2.1). This was also motivated by the fact that several theories satisfy the above property. For instance the universal (and thus the complete theory) of a linear group, over a commutative noetherian ring, e.g, a field, has at most \aleph_0 nonisomorphic finitely generated models. More generally, the universal theory of an equationally noetherian group satisfies the same property [OH05]. We note that linear groups over a commutative noetherian ring, e.g, a field, are equationally noetherian and not all equationally noetherian groups are linear [BMR99].

Given a theory T , we consider the following type, in the language $L(\bar{c}) = L \cup \{c_1, \dots, c_n\}$, where c_1, \dots, c_n are a new constants symbols,

$$p_n(y) = \{y \neq \tau(\bar{c}) \mid \tau(\bar{x}) \text{ a term in } L\}.$$

Then a model (\mathcal{M}, \bar{a}) of T , in the language $L(\bar{c})$, is generated by \bar{a} if and only (\mathcal{M}, \bar{a}) omits p_n . Thus the class of n -generated models of T is the class of models of T , in the language $L(\bar{c})$, which omit p_n . We adopt this viewpoint, and for that reason we need to omit some types in the class of models omitting types including p_n (Theorem 3.1 and Theorem 5.1).

We begin by showing, in the next section, that the number $\alpha(T)$ of nonisomorphic finitely generated models of T is at most \aleph_0 or equals to 2^{\aleph_0} . This result is analogue to the one known on the number of complete types of countable theories. Our aim in section 3 is to define a rank of n -generated models of T , and to prove that when T has at most countably many nonisomorphic n -generated models, every n -generated model of T has an ordinal rank. As any n -generated model of T is determined by its complete type relatively to some n -generating tuple, the previous result can be seen as an analogue of the Morley rank of types of an ω -stable theory T . Nevertheless, the rank that we define is different from the Morley rank, since it uses actually the class of n -generated models of T and T is not necessarily ω -stable.

In section 4 we turn to use the rank to give a property of n -generated models of countable theories, having at most countably many nonisomorphic finitely generated models, which can be seen as an analogue to the Hopf property of groups (Theorem 4.3). Recall that a group G is said to be *Hopfian*, if any surjective morphism from G to G is an isomorphism. It is known that finitely generated linear groups are Hopfian [LS77]. More generally, finitely generated equationally noetherian groups are Hopfian [OH05]. The property obtained for finitely generated models is as follows. For every generating n -tuple \bar{a} of \mathcal{M} , there exists a formula $\phi(\bar{x})$ such that $\mathcal{M} \models \phi(\bar{a})$ and such that for any generating n -tuple \bar{b} of \mathcal{M} , $(\mathcal{M}, \bar{a}) \cong (\mathcal{M}, \bar{b})$ if and only if $\mathcal{M} \models \phi(\bar{b})$. In the same section, we use this to get a necessary and sufficient condition for a finitely generated model to be prime of its complete theory (Theorem 4.5).

When T is universal, we define an another rank, more easy to use. This will be done in section 4. Since the universal theory of an equationally noetherian group H has at most countably many nonisomorphic finitely generated models, one can attribute a rank to H -limit groups and study them in this context. We give some results in this direction in section 5.

2. THE NUMBER OF FINITELY GENERATED MODELS.

For a theory T we denote by $\alpha_n(T)$ (resp. $\alpha(T)$) the number of nonisomorphic n -generated (resp. finitely generated) models of T .

Theorem 2.1. *For every theory T , $\alpha_n(T) \leq \aleph_0$ or $\alpha_n(T) = 2^{\aleph_0}$. Therefore $\alpha(T) \leq \aleph_0$ or $\alpha(T) = 2^{\aleph_0}$.*

Before proving the theorem, we need some notions. Let Φ be a sentence of $L_{\omega_1\omega}$ and L_A be a countable fragment of $L_{\omega_1\omega}$. We define an L_A - n -type to be a set p such that

$$p = \{\varphi(\bar{x}) \in L_A \mid \mathcal{M} \models \varphi(\bar{a})\},$$

for some model \mathcal{M} of Φ and some tuple $\bar{a} \in \mathcal{M}^n$. The set of all L_A - n -types of Φ is denoted by $S_n(L_A, \Phi)$. For more details, the reader is referred to [Mar02, Mor70].

Proposition 2.2. [Mor70, Corollary 2.4] *For every countable fragment L_A the set $S_n(L_A, \Phi)$ is either countable or of power 2^{\aleph_0} .* \square

Proof of Theorem 2.1

Let

$$\Phi = \bigwedge_{\varphi \in T} \varphi \wedge \exists \bar{x} \forall y \left(\bigvee_{\tau \in Ter} y = \tau(\bar{x}) \right),$$

where Ter is the set of all terms of L with free variables among \bar{x} , and the length of \bar{x} is n . Then $\Phi \in L_{\omega_1\omega}$ and a model of T is n -generated if and only if it satisfies Φ . Let L_A be the set of all formulas of L . Then L_A is a countable fragment and by Proposition 2.2, $S_n(L_A, \Phi)$ is countable or of power 2^{\aleph_0} .

If $S_n(L_A, \Phi)$ is countable, then Φ has at most \aleph_0 models; as any complete type of a model of Φ , respectively to some generating n -tuple, is an L_A - n -type. Thus T has at most countably many nonisomorphic n -generated models.

If $S_n(L_A, \Phi)$ is of power 2^{\aleph_0} , then Φ has 2^{\aleph_0} models (and thus T has 2^{\aleph_0} n -generated models), as any model of Φ realizes at most a countable number of L_A - n -types. \square

There are several examples of theories having at most \aleph_0 nonisomorphic finitely generated models. For example abelian group theory, complete theory of finitely generated linear groups, and more generally the complete theory (or the universal theory) of a submodel of an ω -stable model. In particular, the universal theory of non-abelian free groups has \aleph_0 nonisomorphic finitely generated models, as every non-abelian free group is a subgroup of $GL_n(\mathbb{C})$ and this last group has a finite Morley rank. This is also a consequence, as noticed in the introduction, of the fact that linear groups are equationally noetherian. We regroup this remarks in the following proposition.

Proposition 2.3. *Let T be a countable ω -stable theory and \mathcal{M} a model of T . Then the universal theory of any submodel of \mathcal{M} has at most \aleph_0 nonisomorphic finitely generated models.*

Proof. Let $\mathcal{N} \subseteq \mathcal{M}$ and denote by Γ the universal theory of \mathcal{N} . Suppose towards a contradiction that $\alpha(\Gamma) > \aleph_0$ and thus $\alpha_n(\Gamma) > \aleph_0$ for some $n \in \mathbb{N}^*$. Then, for any n -finitely generated model \mathcal{A} of Γ , generated by \bar{a} , the type

$$p_{\bar{a}}(\mathcal{A}) = \{\phi(\bar{x}) \mid \phi \text{ is atomic or negatomic such that } \mathcal{A} \models \phi(\bar{a})\}$$

is a consistent type with the universal theory of \mathcal{M} . By compactness, there exists a model of $\text{Th}(\mathcal{M})$ which contains a copy of every n -generated model of Γ . Therefore, as there exists $\alpha_n(\Gamma) > \aleph_0$ nonisomorphic n -generated models of Γ , $\text{Th}(\mathcal{M})$ has more than \aleph_0 types on \emptyset . A contradiction with the ω -stability of \mathcal{M} . \square

One can extracted from papers of F.Oger [Oge82, Oge91, Oge98] that for every $n \in \mathbb{N}^*$, there exists a complete theory T of groups such that $\alpha(T) = n$. G. Sabbagh asked the following.

Problem. Is there a complete theory T of groups such that $\alpha(T) = 2^{\aleph_0}$?

3. A RANK

We shall define a *rank* of n -generated models of T . Before proceeding, we need some notions around omitting types in some classes of models. For our purpose, a type is a set of sentences in the language $L(\bar{x})$. Let \mathcal{K} be a class of models. Given a type q , we say that q is *supported over* \mathcal{K} if there exists a formula $\phi(\bar{x})$ such that:

- (i) some model in \mathcal{K} has a tuple satisfying ϕ , and
- (ii) in every model in \mathcal{K} , each tuple satisfying ϕ realizes q ,

and in that case we say that ϕ *supports* q over \mathcal{K} . We say that q is *unsupported* over \mathcal{K} if it is not supported over \mathcal{K} .

Let T be a theory in L and P a set of types. We denote by $\mathcal{K}(T|P)$ the class of models of T omitting every p in P . The next theorem is a slight refinement of the classical omitting types theorem.

Theorem 3.1. *Let P and Q be a countable sets of types. If $\mathcal{K}(T|P)$ is not empty and each $q \in Q$ is unsupported over $\mathcal{K}(T|P)$, then there exists a countable model in $\mathcal{K}(T|P)$ which omits every $q \in Q$.*

Proof. Let T' be the set of all sentences of L which are true in every model in $\mathcal{K}(T|P)$. We claim that over T' , all types of P and all types of Q are with no support; that is, they are unsupported over the class of models of T' .

Let $p \in P$ and suppose that ϕ supports p over T' , and let \mathcal{M} be any model of T omitting all types of P . Then since \mathcal{M} is a model of T' , every element satisfying ϕ must realizes p ; hence no element of \mathcal{M} satisfies ϕ . So $\neg\exists x\phi(x)$ is in T' . Contradiction.

Clearly if $q \in Q$ is supported over T' then q is supported over $\mathcal{K}(T|P)$. Therefore, by the standard omitting types theorem, there is a countable model \mathcal{M} of T' omitting all types of P and all types of Q . Since $T \subseteq T'$, \mathcal{M} is a model of T and thus \mathcal{M} is in $\mathcal{K}(T|P)$ and it omits every $q \in Q$. \square

We note that to derive the standard omitting theorem from the above theorem, it is sufficient to take P to be the empty set.

Let $L_n(\bar{c}) = L \cup \{\bar{c}\}$, where \bar{c} is a new n -tuple of constants symbols. For an n -generated model \mathcal{M} of T , generated by \bar{a} , we let

$$p_{\bar{a}}(\mathcal{M}) = \{\phi(\bar{c}) \mid \phi \text{ is formula such that } \mathcal{M} \models \phi(\bar{a})\},$$

in the language $L_n(\bar{c})$. We let, as in the introduction,

$$p_n(y) = \{y \neq \tau(\bar{c}) \mid \tau(\bar{x}) \text{ is a term in } L, |\bar{x}| = n\}.$$

As noticed in the introduction, a model (\mathcal{M}, \bar{a}) of T , in the language $L_n(\bar{c})$, is generated by \bar{a} if and only (\mathcal{M}, \bar{a}) omits p_n .

Definition 3.2. Let \mathcal{M} be an n -generated model of T . We define inductively $rk_n(\mathcal{M}) = \alpha$ for an ordinal $\alpha \geq 1$ as follows.

- $rk_n(\mathcal{M}) = 1$ if and only if for every generating n -tuple \bar{a} of \mathcal{M} , $p_{\bar{a}}(\mathcal{M})$ is supported over $\mathcal{K}(T|P_{n,1})$, where $P_{n,1} = \{p_n\}$.
- $rk_n(\mathcal{M}) = \alpha$ if and only if for every generating n -tuple \bar{a} of \mathcal{M} , $p_{\bar{a}}(\mathcal{M})$ is supported over $\mathcal{K}(T|P_{n,\alpha})$, where for $\alpha \geq 2$,

$$P_{n,\alpha} = \{p_{0,n}\} \cup \{p_{\bar{b}}(\mathcal{N}) \mid \mathcal{N} \models T, \bar{b} \text{ generates } \mathcal{N}, |\bar{b}| = n, rk_n(\mathcal{N}) < \alpha\}.$$

- $rk_n(\mathcal{M}) = \infty$ if there is no ordinal α such that $rk_n(\mathcal{M}) = \alpha$.

It should be noted that we work in the language $L_n(\bar{c})$.

We first show that this rank does not depends on n .

Proposition 3.3. *Let T be a theory in L . If \mathcal{M} is an n -generated model of T , which is also m -generated, then $rk_n(\mathcal{M}) = rk_m(\mathcal{M})$.*

Proof. Let $n, m \in \mathbb{N}^*$. We prove by induction on $\gamma \geq 1$, that if \mathcal{M} is an n -generated of T , which is also m -generated, and $rk_n(\mathcal{M}) = \gamma$, then $rk_m(\mathcal{M}) = \gamma$.

Let \mathcal{M} be an n -generated model of T , which is also m -generated and suppose that $rk_n(\mathcal{M}) = \gamma$. Then for any generating n -tuple \bar{a} of G , there exists a sentence

$\phi(\bar{c})$, with $|\bar{c}| = n$, such that for any n -generated model \mathcal{A} of T generated by the n -tuple \bar{h} , if $\mathcal{A} \models \phi(\bar{h})$ and $(\mathcal{A}, \bar{h}) \in \mathcal{K}(T|P_{n,\gamma})$, then $(\mathcal{M}, \bar{a}) \cong (\mathcal{A}, \bar{h})$.

Let \bar{b} be an m -tuple which generates \mathcal{M} . Then there exist an n -tuple \bar{t} of terms such that $\bar{a} = \bar{t}(\bar{b})$ in \mathcal{M} , and an m -tuple $\bar{\tau}$ of terms such that $\bar{b} = \bar{\tau}(\bar{a})$ in \mathcal{M} . Let

$$\psi(\bar{y}) = \exists \bar{x}(\bar{x} = \bar{t}(\bar{y}) \wedge \bar{y} = \bar{\tau}(\bar{x}) \wedge \phi(\bar{x})).$$

Then $\mathcal{M} \models \psi(\bar{b})$.

Suppose first $\gamma = 1$. We claim that $\psi(\bar{y})$ supports $p_{\bar{b}}(\mathcal{M})$ over $\mathcal{K}(T|P_{m,1})$. Let \mathcal{A} be a model of T , generated by the m -tuple \bar{d} , such that $\mathcal{A} \models \psi(\bar{d})$. Since $\mathcal{A} \models \psi(\bar{d})$, there exists an n -tuple \bar{h} , which generates \mathcal{A} , such that $\mathcal{A} \models \phi(\bar{h})$. Then $(\mathcal{A}, \bar{h}) \cong (\mathcal{M}, \bar{a})$. Therefore $(\mathcal{A}, \bar{d}) \cong (\mathcal{M}, \bar{b})$, as $\mathcal{A} \models \bar{d} = \bar{\tau}(\bar{h})$ and $\mathcal{A} \models \bar{b} = \bar{\tau}(\bar{a})$. Hence $rk_m(\mathcal{M}) = 1$, and this ends the proof of our claim and the proof in the case $\gamma = 1$.

Now suppose that $rk_n(\mathcal{M}) = \gamma \geq 2$ and that for any ordinal $1 \leq \delta < \gamma$ and for any n -generated model \mathcal{N} of T , if \mathcal{N} is m -generated and if $rk_n(\mathcal{N}) = \delta$ then $rk_m(\mathcal{N}) = \delta$.

We claim that $\psi(\bar{y})$ supports $p_{\bar{b}}(\mathcal{M})$ over $\mathcal{K}(T|P_{m,\gamma})$. Let \mathcal{A} be a model of T , generated by the m -tuple \bar{d} , such that $\mathcal{A} \models \psi(\bar{d})$ and $(\mathcal{A}, \bar{d}) \in \mathcal{K}(T|P_{m,\gamma})$, and let us prove that $(\mathcal{M}, \bar{b}) \cong (\mathcal{A}, \bar{d})$.

As before, there exists an n -tuple \bar{h} , which generates \mathcal{A} , such that $\mathcal{A} \models \phi(\bar{h})$. Let us prove that $(\mathcal{A}, \bar{h}) \in \mathcal{K}(T|P_{n,\gamma})$. Suppose towards a contradiction, that $(\mathcal{A}, \bar{h}) \notin \mathcal{K}(T|P_{n,\gamma})$ and thus (\mathcal{A}, \bar{h}) realizes $p_{\bar{f}}(\mathcal{N})$ for some $\mathcal{N} \models T$ and $rk_n(\mathcal{N}) = \delta < \gamma$. Then $(\mathcal{A}, \bar{h}) \cong (\mathcal{N}, \bar{f})$ and thus $rk_n(\mathcal{A}) = rk_n(\mathcal{N}) = \delta < \gamma$. Therefore, by induction, $rk_m(\mathcal{A}) = rk_n(\mathcal{A}) < \gamma$; as \mathcal{A} is m -generated. A contradiction as $(\mathcal{A}, \bar{d}) \in \mathcal{K}(T|P_{m,\gamma})$. Therefore $(\mathcal{A}, \bar{h}) \in \mathcal{K}(T|P_{n,\gamma})$.

Since $(\mathcal{A}, \bar{h}) \in \mathcal{K}(T|P_{n,\gamma})$ and $\mathcal{A} \models \psi(\bar{d})$, $(\mathcal{A}, \bar{h}) \cong (\mathcal{M}, \bar{a})$. Therefore $(\mathcal{A}, \bar{d}) \cong (\mathcal{M}, \bar{b})$, as $\mathcal{A} \models \bar{d} = \bar{\tau}(\bar{h})$ and $\mathcal{A} \models \bar{b} = \bar{\tau}(\bar{a})$. Thus $\psi(\bar{y})$ supports $p_{\bar{b}}(\mathcal{M})$ over $\mathcal{K}(T|P_{m,\gamma})$ as claimed. Hence $rk_m(\mathcal{M}) = \gamma$. This ends the proof of the induction.

Now if \mathcal{M} is an n -generated model of T , which is also m -generated, with $rk_n(\mathcal{M}) = \infty$, then by the precedent result, we deduce $rk_m(\mathcal{M}) = \infty$. \square

Proposition 3.3 allows us to define the rank of \mathcal{M} in a way that does not depends on the length of generating tuple of \mathcal{M} . So we let $rk(\mathcal{M})$ to be $rk_n(\mathcal{M})$ if \mathcal{M} is n -generated. When $\alpha_n(T) \leq \aleph_0$, we have a good characterization.

Theorem 3.4. *If T is an n -consistent theory satisfying $\alpha_n(T) \leq \aleph_0$, then every n -generated model of T has an ordinal rank.*

Lemma 3.5. *Let \mathcal{M} be an n -generated model of T , generated by the n -tuple \bar{a} , such that $p_{\bar{a}}(\mathcal{M})$ is supported over $\mathcal{K}(T|P_{n,\gamma})$. Then for every generating n -tuple \bar{b} of \mathcal{M} the type $p_{\bar{b}}(\mathcal{M})$ is also supported over $\mathcal{K}(T|P_{n,\gamma})$.*

Proof. There is a finite n -tuple of terms $\bar{\tau}(\bar{x})$ such that $\mathcal{M} \models \bar{b} = \bar{\tau}(\bar{a})$. Let $\phi(\bar{c})$ be a sentence supporting $p_{\bar{a}}(\mathcal{M})$ over $\mathcal{K}(T|P_{n,\gamma})$, and Let $\psi(\bar{c}) = \exists \bar{x}(\bar{c} = \bar{\tau}(\bar{x})) \wedge \phi(\bar{x})$. We claim that $\psi(\bar{c})$ supports $p_{\bar{b}}(\mathcal{M})$ over $\mathcal{K}(T|P_{n,\gamma})$. Clearly $(\mathcal{M}, \bar{a}) \in \mathcal{K}(T|P_{n,\gamma})$ and $\mathcal{M} \models \psi(\bar{b})$.

Let $(\mathcal{N}, \bar{h}) \in \mathcal{K}(T|P_{n,\gamma})$ such that $\mathcal{N} \models \psi(\bar{h})$. Then, there exists \bar{d} such that $\mathcal{N} \models \phi(\bar{d})$ and $\mathcal{N} \models (\bar{h} = \bar{\tau}(\bar{d}))$. Therefore, \mathcal{N} is generated by \bar{d} . So (\mathcal{N}, \bar{d}) omits $p_{\bar{a}}$, and clearly (\mathcal{N}, \bar{d}) is in $\mathcal{K}(T|P_{n,\gamma})$. Therefore, as $\mathcal{N} \models \phi(\bar{d})$, $(\mathcal{M}, \bar{a}) \cong (\mathcal{N}, \bar{d})$. Hence, $(\mathcal{M}, \bar{b}) \cong (\mathcal{N}, \bar{h})$, as $\mathcal{M} \models \bar{\tau}(\bar{a}) = \bar{b}$ and $\mathcal{N} \models \bar{\tau}(\bar{d}) = \bar{h}$. Thus (\mathcal{N}, \bar{h}) realizes $p_{\bar{b}}(\mathcal{M})$. \square

Proof of Theorem 3.4.

Let γ be the least ordinal such that if \mathcal{M} is an n -generated model of T of ordinal rank, then $rk(\mathcal{M}) < \gamma$. Suppose towards a contradiction that there exists an n -generated model \mathcal{N} of T such that $rk(\mathcal{N}) = \infty$. We are going to prove that there exists an n -generated model \mathcal{M} of T of rank γ and thus we get a contradiction.

Since $\alpha_n(T) \leq \aleph_0$, $P_{n,\gamma}$ is countable. Notice that every n -generated model of T is either in $\mathcal{K}(T|P_{n,\gamma})$ or it has an ordinal rank.

By our supposition above, for every generating n -tuple \bar{a} of \mathcal{N} , the model (\mathcal{N}, \bar{a}) omits every p in $P_{n,\gamma}$. Hence $\mathcal{K}(T|P_{n,\gamma})$ is not empty. Now we prove the following claim.

Claim. There exists an n -generated model \mathcal{M} of T , generated by an n -tuple \bar{a} , such that $p_{\bar{a}}(\mathcal{M})$ is supported over $\mathcal{K}(T|P)$.

Proof. Since $\alpha_n(T) \leq \aleph_0$, there exists at most \aleph_0 nonisomorphic n -generated models of T which have an infinite rank. Let $((\mathcal{M}_i, \bar{a}_i), \bar{a}_i \text{ generates } \mathcal{M}_i : i \in \beta \leq \aleph_0)$ be the list of nonisomorphic n -generated models of T , such that $rk(\mathcal{M}_i) = \infty$.

Suppose that for every $i \in \beta$, $p_{\bar{a}_i}(\mathcal{M}_i)$ is unsupported over $\mathcal{K}(T|P_{n,\gamma})$. Then by Theorem 3.1, there exists a model (\mathcal{N}, \bar{b}) of T in $\mathcal{K}(T|P_{n,\gamma})$ which omits $p_{\bar{a}_i}(\mathcal{M}_i)$ for every $i \in \beta$.

Therefore $\mathcal{N} \not\cong \mathcal{M}_i$ for every $i \in \beta$, and since (\mathcal{N}, \bar{b}) omits every $p \in P_{n,\gamma}$ we have $rk(\mathcal{N}) = \infty$. A contradiction.

Hence there exists $\ell \in \beta$ such that $p_{\bar{a}_\ell}(\mathcal{M}_\ell)$ is supported over $\mathcal{K}(T|P_{n,\gamma})$. \square

By Lemma 3.5 and by the Claim above, there exists an n -generated model \mathcal{M} of T , such that $rk(\mathcal{M}) = \gamma$. A final contradiction. \square

Remark. It should be remarked that $\mathcal{M} \in \mathcal{K}(T|P_{n,\gamma})$ if and only if $rk(\mathcal{M}) \geq \gamma$. This property will be used freely without any reference to it.

Examples.

(1). Let Γ be the universal theory of torsion-free abelian groups. Then every finitely generated model of Γ is free abelian of finite rank. In fact, it is well known that for any $n \in \mathbb{N}^*$, every finitely generated group which satisfies $Th(\mathbb{Z}^n)$ is isomorphic to \mathbb{Z}^n . Therefore for every free abelian group G of finite rank n , we have $rk(G) = 1$, relatively to $Th(\mathbb{Z}^n)$. We will see that the rank of nontrivial free abelian groups coincide with another rank Rk defined for universal theories, relatively to Γ . (section 5).

(2). If G is a finitely generated abelian group and if we let Γ to be the complete theory of G , then every finitely generated model of Γ is isomorphic to G and thus $\alpha(\Gamma) = rk(G) = 1$. According to [Nie03], a finitely generated group G is said *quasi-finitely axiomatisable* (abbreviated QFA), if there exists a sentence ϕ satisfied by G such that any finitely generated group satisfying ϕ is isomorphic to G . A. Nies [Nie03] proves that the free nilpotent group of class 2 with 2 generators is QFA. F. Oger and G. Sabbagh [OS] generalize this result by showing that any finitely generated free nilpotent group of class ≥ 2 is QFA. Moreover they prove that a finitely generated nilpotent group is QFA if and only if it is prime model of its theory. Thus the complete theory Γ of a finitely generated free nilpotent group of class ≥ 2 satisfies $\alpha(\Gamma) = 1$.

It is useful to notice that in general $\alpha_n(T) \leq \aleph_0$ does not implies $\alpha_m(T) \leq \aleph_0$, for $m \geq n$. For instance the group theory T_{gp} has at most \aleph_0 nonisomorphic 1-generated groups, which are the cyclic groups. However, it is known that T_{gp} has 2^{\aleph_0} nonisomorphic 2-generated groups (see [LS77]).

4. A PROPERTY ANALOGUE TO THE HOPF PROPERTY OF GROUPS

We give in this section some properties of n -generated models of a theory T satisfying $\alpha_n(T) \leq \aleph_0$. One of those properties is analogue to the Hopf property of a group. Recall that a group G is said *Hopfian* or has the *Hopf property* if every surjective morphism from G to G is an isomorphism.

Theorem 4.1. *Let T be an n -consistent theory in L such that $\alpha_n(T) \leq \aleph_0$ and let \mathcal{M} be an n -generated model of T . Then for every generating tuple \bar{a} of \mathcal{M} , there exists a formula $\phi(\bar{x})$ such that for every finitely generated model \mathcal{N} of T , generated by \bar{b} , $(\mathcal{M}, \bar{a}) \cong (\mathcal{N}, \bar{b})$ if and only if $\mathcal{N} \models \phi(\bar{b})$ and $rk(\mathcal{N}) \geq rk(\mathcal{M})$.*

Proof. By Theorem 3.4, $rk(\mathcal{M}) = \gamma$ for some ordinal $\gamma \geq 1$. By the definition of the rank, for every generating m -tuple \bar{a} of \mathcal{M} , $p_{\bar{a}}(\mathcal{M})$ is supported over $\mathcal{K}(T|P_{m,\gamma})$. Therefore, there exists a formula $\phi(\bar{x})$ such that $\mathcal{M} \models \phi(\bar{a})$ and for every m -generated model \mathcal{N} of T , generated by \bar{b} , if $\mathcal{N} \models \phi(\bar{b})$ and $(\mathcal{N}, \bar{b}) \in \mathcal{K}(T|P_{m,\gamma})$ then (\mathcal{N}, \bar{b}) realizes $p_{\bar{a}}(\mathcal{M})$. Thus we get $(\mathcal{M}, \bar{a}) \cong (\mathcal{N}, \bar{b})$ if and only if $\mathcal{N} \models \phi(\bar{b})$. Since $(\mathcal{N}, \bar{b}) \in \mathcal{K}(T|P_{m,\gamma})$ if and only if $rk(\mathcal{N}) \geq \gamma$, we get the desired result. \square

Corollary 4.2. *Let T be an n -consistent theory in L such that $\alpha_n(T) \leq \aleph_0$. Then there exists an n -generated model \mathcal{M} of T , such that for every generating tuple \bar{a} of \mathcal{M} , there exists a formula $\phi(\bar{x})$ such that for every finitely generated model \mathcal{N} of T , generated by \bar{b} , $(\mathcal{M}, \bar{a}) \cong (\mathcal{N}, \bar{b})$ if and only if $\mathcal{N} \models \phi(\bar{b})$.*

Proof. By Theorem 4.1, every n -generated model of T of ordinal rank 1, satisfies the conclusions of the corollary. Let γ be the least ordinal such that there exists an n -generated model of T of ordinal rank γ . Then, by definition of the rank, we have $\gamma = 1$. \square

The following theorem translates an "internal" property of all n -generated models of a theory T having at most \aleph_0 nonisomorphic n -generated models.

Theorem 4.3. *Let T be an n -consistent theory in L such that $\alpha_n(T) \leq \aleph_0$, and let \mathcal{M} be an n -generated model of T . Then for every generating tuple \bar{a} of \mathcal{M} there exists a formula $\phi(\bar{x})$ such that for every generating tuple \bar{b} of \mathcal{M} we have $(\mathcal{M}, \bar{a}) \cong (\mathcal{M}, \bar{b})$ if and only if $\mathcal{M} \models \phi(\bar{b})$.*

Proof. A consequence of Theorem 4.1. \square

A consequence of the above property of \mathcal{M} can be expressed as follows. If \bar{a} is an n -generating tuple of \mathcal{M} and $f : \mathcal{M} \rightarrow \mathcal{M}$ is a surjective morphism such that $\mathcal{M} \models \phi(f(\bar{a}))$, then f is an isomorphism. When G is a finitely generated Hopfian group, the formula $\phi(\bar{x})$ can be taken to be $\bar{x} = \bar{x}$.

We define a finitely generated model \mathcal{M} to be *weak-Hopfian* if for any generating tuple \bar{a} of \mathcal{M} , there exists a formula $\phi(\bar{x})$ such that for any surjective morphism $f : \mathcal{M} \rightarrow \mathcal{M}$ if $\mathcal{M} \models \phi(f(\bar{a}))$, then f is an isomorphism. The next corollary is therefore a consequence of Theorem 4.3.

Corollary 4.4. *Let T be an n -consistent theory in L such that $\alpha(T) \leq \aleph_0$. Then every finitely generated model of T is weak-Hopfian.* \square

A natural question arises in this context. When a finitely generated model can be prime of its complete theory ? The following theorem gives a necessary and sufficient condition.

Theorem 4.5. *Let T be a complete theory such that $\alpha_n(T) \leq \aleph_0$. An n -generated model \mathcal{M} of T is prime if and only if there exists a formula $\theta(\bar{x})$, satisfied by some tuple in \mathcal{M} , such that if $\mathcal{M} \models \theta(\bar{a})$ then \bar{a} generates \mathcal{M} .*

We use the following classical result.

Proposition 4.6. [Hod93] *Let \mathcal{M} be a countable model. Then \mathcal{M} is a prime model of its theory iff for every $m \in \mathbb{N}^*$, each orbit under the action of $\text{Aut}(\mathcal{M})$ on \mathcal{M}^m is first-order definable without parameters.*

Proof of Theorem 4.5

Suppose that \mathcal{M} is a prime model of its theory and let \bar{a} generates \mathcal{M} . Then there is some orbit \mathcal{O}_n containing \bar{a} . By Proposition 4.6, \mathcal{O}_n is definable by a first order formula $\theta(\bar{x})$. Now if $\mathcal{M} \models \theta(\bar{b})$, then there is an automorphism f such that $f(\bar{a}) = \bar{b}$, and therefore \bar{b} generates \mathcal{M} .

Suppose now that there exists a formula $\theta(\bar{x})$ consistent in \mathcal{M} such that if $\mathcal{M} \models \theta(\bar{a})$ then \bar{a} generates \mathcal{M} . Let \bar{a} in \mathcal{M} such that $\mathcal{M} \models \theta(\bar{a})$. Then by Theorem 4.3, there exists a sentence $\phi(\bar{c})$ in $L(\bar{c})$ such that $\mathcal{M} \models \phi(\bar{a})$, and if \bar{b} generates \mathcal{M} such that $\mathcal{M} \models \phi(\bar{b})$, then the function defined by $f(\bar{a}) = \bar{b}$ extends to an automorphism.

Let \mathcal{O}_m be an orbit and let \bar{t} be an m -tuple of terms such that $\bar{t}_1(\bar{a}) \in \mathcal{O}_m$. Let us show that \mathcal{O}_m is defined by the formula

$$\psi(\bar{y}) = \exists \bar{z}(\phi(\bar{z}) \wedge \theta(\bar{z}) \wedge \bar{y} = \bar{t}(\bar{z})).$$

Let $\bar{b} \in \mathcal{O}_m$. Then there is an automorphism f such that $f(\bar{t}(\bar{a})) = \bar{b}$. Therefore $\bar{t}(f(\bar{a})) = \bar{b}$, and $\mathcal{M} \models \phi(f(\bar{a})) \wedge \theta(f(\bar{a}))$. Thus $\mathcal{M} \models \psi(\bar{b})$.

Now let $\bar{b} \in \mathcal{M}$ such that $\mathcal{M} \models \psi(\bar{b})$. Then there is tuple \bar{d} in \mathcal{M} such that $\mathcal{M} \models \phi(\bar{d}) \wedge \theta(\bar{d}) \wedge \bar{b} = \bar{t}(\bar{d})$. Hence \bar{d} generates \mathcal{M} and since $\mathcal{M} \models \phi(\bar{d})$ there is an automorphism f such that $f(\bar{a}) = \bar{d}$. Therefore $f(\bar{t}(\bar{a})) = \bar{t}(\bar{d}) = \bar{b}$. Thus $\bar{b} \in \mathcal{O}_m$. \square

Definition 4.7. A theory T is said to be n -categorical if $\alpha_n(T) = 1$.

Examples

(1) Let F_2 be the free non-abelian group on two generators. Then $Th_{\forall\exists}(F_2)$ is 2-categorical. In fact, $Th_{\forall}(F_2) \cup \{\exists x \exists y([x, y] \neq 1)\}$ is 2-categorical. Indeed, if A is a model of $Th_{\forall}(F_2)$ generated by $\{a, b\}$ and $[a, b] \neq 1$, then it is well-known that $\{a, b\}$ generates a free group with basis $\{a, b\}$.

(2) For every n , $Th(\mathbb{Z}^n)$ is m -categorical for every m .

Corollary 4.8. *Let T be a complete n -consistent theory satisfying $\alpha_n(T) \leq \aleph_0$. Then the following properties are equivalents*

(1) $\alpha(T) = 1$ and the unique finitely generated model of T is prime.

(2) There exists a formula $\theta(\bar{x})$, consistent with T such that: for every finitely generated model \mathcal{M} of T and for every \bar{b} in \mathcal{M} if $\mathcal{M} \models \theta(\bar{b})$ then \bar{b} generates \mathcal{M} . \square

5. THE SPECIAL CASE OF UNIVERSAL THEORIES

In this section we define another rank specific to universal theories. In fact this rank can also be defined for $\forall\exists$ -theories and therefore we work in this context. Before proceeding, we need some adaptation of Theorem 3.1 and Theorem 5.1 to our context.

Let \mathcal{K} be a class of models. For an *universal* type q , we say that q is *existentially supported over \mathcal{K}* if there exists an *existential* formula $\phi(\bar{x})$ which supports q over \mathcal{K} . We say that q is *existentially unsupported over \mathcal{K}* if it is not existentially supported over \mathcal{K} .

Let T be a $\forall\exists$ -theory in L and P a set of universal types. The next theorem is an adaptation of Theorem 3.1. The proof is very similar to the proof of that theorem.

Theorem 5.1. *Let P and Q be a countable sets of universal types. If $\mathcal{K}(T|P)$ is not empty and each $q \in Q$ is existentially unsupported over $\mathcal{K}(T|P)$, then there exists a countable model in $\mathcal{K}(T|P)$ which omits every $q \in Q$.*

Proof. Let T' be the set of all $\forall\exists$ -sentences of L which are true in every model in $\mathcal{K}(T|P)$. We claim that over T' , all types of P and all types of Q are with no existential support; i.e they are existentially unsupported over the class of models of T' .

Let $p \in P$ and suppose that ϕ is existential and supports p over T' , and let \mathcal{M} be any model of T omitting all types of P . Then since \mathcal{M} is a model of T' , every element satisfying ϕ must realizes p ; hence no element of \mathcal{M} satisfies ϕ . So $\neg\exists\bar{x}\phi(\bar{x})$, which is an universal sentence, is in T' . Contradiction.

Clearly if $q \in Q$ is existential supported over T' then q is existentially supported over $\mathcal{K}(T|P)$. Therefore, by the omitting types theorem for $\forall\exists$ -theories (see for instance [CK73] or [Hod93]), there is a countable existentially closed model \mathcal{M} of T' omitting all types of P and all types of Q . Since $T \subseteq T'$, \mathcal{M} is a countable model of T and thus \mathcal{M} is in $\mathcal{K}(T|P)$ and it omits every $q \in Q$. \square

If \mathcal{M} is a model of T , generated by the n -tuple \bar{a} , we let

$$p_{\bar{a},\forall}(\mathcal{M}) = \{\psi(\bar{c}) \mid \phi(\bar{x}) \text{ is universal, } \mathcal{M} \models \phi(\bar{a})\}.$$

Definition 5.2. Let \mathcal{M} be an n -generated model of T . We define inductively $Rk_n(\mathcal{M}) = \alpha$ for an ordinal $\alpha \geq 1$ as follows.

- $Rk_n(\mathcal{M}) = 1$ if and only if for every generating n -tuple \bar{a} of \mathcal{M} , $p_{\bar{a}}(\mathcal{M})$ is supported over $\mathcal{K}(T|Q_{n,1})$, where $Q_{n,1} = \{p_n\}$.
- $Rk_n(\mathcal{M}) = \alpha$ if and only if for every generating n -tuple \bar{a} of \mathcal{M} , $p_{\bar{a}}(\mathcal{M})$ is supported over $\mathcal{K}(T|Q_{n,\alpha})$, where for $\alpha \geq 2$

$$Q_{n,\alpha} = \{p_{0,n}\} \cup \{p_{\bar{b},\forall}(\mathcal{N}) \mid \mathcal{N} \models T, \bar{b} \text{ generates } \mathcal{N}, |\bar{b}| = n, Rk_n(\mathcal{N}) < \alpha\}.$$

- $Rk_n(\mathcal{M}) = \infty$ if there is no ordinal α such that $Rk_n(\mathcal{M}) = \alpha$.

As in section 3, this rank does not depends on n and we denote it Rk .

Using Theorem 5.1, the proof of the following theorem is just an adaptation of the proof of Theorem 3.4.

Theorem 5.3. *If T is an n -consistent $\forall\exists$ -theory satisfying $\alpha_n(T) \leq \aleph_0$, then every n -generated model of T has an ordinal rank.* \square

A natural problem in this context is to explicit the relation between the two ranks rk and Rk .

Proposition 5.4. *If T is an n -consistent $\forall\exists$ -theory, then for every n -generated model \mathcal{M} of T , $rk(\mathcal{M}) \leq Rk(\mathcal{M})$.*

Proof.

We prove by induction on $\gamma \geq 1$, that if $Rk(\mathcal{M}) = \gamma$ then $rk(\mathcal{M}) \leq \gamma$.

For $\gamma = 1$ the result follows from the definition of the two ranks.

Now suppose that for every n -generated model \mathcal{N} , if $Rk(\mathcal{N}) = \alpha < \gamma$ then $rk(\mathcal{N}) \leq \alpha$ and let \mathcal{M} be an n -generated model of T with $Rk(\mathcal{M}) = \gamma$.

If $rk(\mathcal{M}) < \gamma$ we get the searched result. So we suppose $rk(\mathcal{M}) \geq \gamma$ and we show that $rk(\mathcal{M}) = \gamma$.

Let \bar{a} be an n -generating tuple of \mathcal{M} . Then there exists a formula $\phi(\bar{c})$ which supports $p_{\bar{a}, \forall}(\mathcal{M})$ over $\mathcal{K}(T|Q_{n, \gamma})$. We claim that $\phi(\bar{c})$ supports $p_{\bar{a}}(\mathcal{M})$ over $\mathcal{K}(T|P_{n, \gamma})$.

Let \mathcal{N} be an n -generated model of T , generated by \bar{b} , such that $\mathcal{N} \models \phi(\bar{b})$ and $rk(\mathcal{N}) \geq \gamma$. Then $Rk(\mathcal{N}) \geq \gamma$; otherwise if $Rk(\mathcal{N}) < \gamma$, by induction $rk(\mathcal{N}) < \gamma$, a contradiction. Thus (\mathcal{N}, \bar{b}) omits every $p \in Q_{n, \gamma}$ and hence $(\mathcal{N}, \bar{b}) \cong (\mathcal{M}, \bar{a})$.

We conclude that if \mathcal{N} is an n -generated model of T , generated by \bar{b} , such that $\mathcal{N} \models \phi(\bar{b})$ and $rk(\mathcal{N}) \geq \gamma$, then $(\mathcal{N}, \bar{b}) \cong (\mathcal{M}, \bar{a})$. As $rk(\mathcal{M}) \geq \gamma$, $\mathcal{K}(T|P_{n, \gamma})$ is not empty and therefore $\phi(\bar{c})$ supports $p_{\bar{a}}(\mathcal{M})$ over $\mathcal{K}(T|P_{n, \gamma})$ as claimed. Thus $rk(\mathcal{M}) = \gamma$. This ends the proof of the induction.

If $Rk(\mathcal{M}) = \infty$ the conclusion is clear. \square

Remark. It is not always the case that $rk(\mathcal{M}) = Rk(\mathcal{M})$. Let, as in the end of section 3, Γ be the universal theory of torsion-free abelian groups. Then every finitely generated model of Γ is free abelian of finite rank. This rank coincide with the above rank; that is, a nontrivial finitely generated torsion-free abelian group G is of rank n if and only if $Rk(G) = n$. This shows that, in general, $rk(\mathcal{M}) \neq Rk(\mathcal{M})$.

The following theorem will be used in the next section.

Theorem 5.5. *Let T be an n -consistent $\forall\exists$ -theory satisfying $\alpha_n(T) \leq \aleph_0$ and let \mathcal{M} be an n -generated model of T , generated by \bar{a} . Then there exists a quantifier-free formula $\phi(\bar{x})$ such that if \mathcal{N} is a finitely generated model of T , generated by \bar{b} , then $(\mathcal{M}, \bar{a}) \cong (\mathcal{N}, \bar{b})$ if and only if $\mathcal{N} \models \phi(\bar{b})$ and $Rk(\mathcal{N}) \geq Rk(\mathcal{M})$.*

Proof. Let $Rk(\mathcal{M}) = \gamma$. By the definition of the rank, for every generating n -tuple \bar{a} of \mathcal{M} , $p_{\bar{a}, \forall}(\mathcal{M})$ is supported over $\mathcal{K}(T|Q_{n, \gamma})$. Therefore, there exists an existential formula $\phi(\bar{x})$ such that $\mathcal{M} \models \phi(\bar{a})$ and for every n -generated model \mathcal{N} of T , generated by \bar{b} , if $\mathcal{N} \models \phi(\bar{b})$ and $(\mathcal{N}, \bar{b}) \in \mathcal{K}(T|Q_{n, \gamma})$ then (\mathcal{N}, \bar{b}) realizes $p_{\bar{a}, \forall}(\mathcal{M})$.

Put $\phi(\bar{x}) = \exists \bar{y} \psi(\bar{x}, \bar{y})$, where $\psi(\bar{x})$ is quantifier-free. Since $\mathcal{M} \models \psi(\bar{a})$, there exists a tuple \bar{d} such that $\mathcal{M} \models \psi(\bar{a}, \bar{d})$. Then there exists a tuple of terms $\bar{t}(\bar{x})$ such that $\mathcal{M} \models \bar{d} = \bar{t}(\bar{a})$. We let

$$\xi(\bar{x}) = \psi(\bar{x}, \bar{t}(\bar{x})).$$

Thus we get $(\mathcal{M}, \bar{a}) \cong (\mathcal{N}, \bar{b})$ if and only if $\mathcal{N} \models \xi(\bar{a})$. Since $(\mathcal{N}, \bar{b}) \in \mathcal{K}(T|Q_{n, \gamma})$ if and only if $Rk(\mathcal{N}) \geq \gamma$, we get the desired result. \square

6. LIMIT GROUPS OF EQUATIONALLY NOETHERIAN GROUPS

In this section we discuss properties of limit groups of equationally noetherian groups, related to their rank.

We begin by recalling some definitions. Let G be a fixed group and $\bar{x} = (x_1, \dots, x_n)$. We denote by $G[\bar{x}]$ the group $G * F(\bar{x})$ where $F(\bar{x})$ is the free group with basis $\{x_1, \dots, x_n\}$. For an element $s(\bar{x}) \in G[\bar{x}]$ and a tuple $\bar{g} = (g_1, \dots, g_n) \in G^n$ we denote by $s(\bar{g})$ the element of G obtained by replacing each x_i by g_i ($1 \leq i \leq n$). Let S be a subset of $G[\bar{x}]$. Then the set

$$V(S) = \{\bar{g} \in G^n \mid s(\bar{g}) = 1 \text{ for all } s \in S\}$$

is termed the *algebraic set* over G defined by S . A group G is called *equationally noetherian* if for every $n \geq 1$ and every subset S of $G[\bar{x}]$ there exists a finite subset $S_0 \subseteq S$ such that $V(S) = V(S_0)$.

Let H be a group. For our purpose, we do not need the exact definition of finitely generated H -limit groups, we use an equivalent definition true when H is equationally noetherian [OH05]. A *finitely generated group* G is said *H -limit* if G is a model of the universal theory of H .

As noticed in the introduction, if H is an equationally noetherian group, then the universal theory of H has at most \aleph_0 nonisomorphic finitely generated models. For completeness we provide a proof of this property.

Throughout this section, if G is a finitely generated group, generated by \bar{a} , we let

$$P(\bar{x}) = \{w(\bar{x}) \mid w \text{ is a word such that } G \models w(\bar{a}) = 1\}.$$

Proposition 6.1. [OH05] *Let H be an equationally noetherian group. Then there exist at most countably many nonisomorphic finitely generated H -limit groups.*

Proof. Suppose towards a contradiction that the opposite is true. Then there exists $n \in \mathbb{N}$ such that there exists at least λ nonisomorphic n -generated H -limit groups for some $\lambda > \aleph_0$. Let $(G_i = \langle \bar{x} | P_i(\bar{x}) \rangle \mid i \in \lambda > \aleph_0)$ be the list of nonisomorphic n -generated H -limit groups. For every $i \in \lambda$ there exists a finite subset $S_i \subseteq P_i$ such that $H \models \forall \bar{x} (S_i(\bar{x}) = 1 \Rightarrow w(\bar{x}) = 1)$ for every $w \in P_i(\bar{x})$.

Since for every $i \in \lambda$ the set S_i is finite, the set $\{S_i \mid i \in \lambda\}$ is countable. Therefore the map $f : \{P_i \mid i \in \lambda\} \rightarrow \{S_i \mid i \in \lambda\}$ defined by $P_i \mapsto S_i$ is not injective and thus there exist $i, j \in \lambda, i \neq j$ such that $S_i = S_j$.

Since G_i, G_j are models of the universal theory of H we get $P_i = P_j$, a contradiction. \square

Therefore, by Theorem 5.3, every finitely generated H -limit group G has an ordinal rank Rk . We are interested on the relation between this rank and decomposition of morphisms from G to another H -limit group.

Theorem 6.2. *Let H be an equationally noetherian group and G be a nontrivial finitely generated H -limit group. Then there exists a finite collection of proper epimorphisms of H -limit groups $(f_i : G \rightarrow L_i \mid 1 \leq i \leq n)$, such that $Rk(L_i) \geq Rk(G)$ whenever L_i is nontrivial, and such that for any H -limit group L , if $Rk(L) \geq Rk(G)$, then any epimorphism $f : G \rightarrow L$, is either an embedding or factors through some f_i .*

Proof. Let \bar{a} be an n -tuple which generates G . Set $\alpha = Rk(G)$. Let $(f_i : G \rightarrow G_i \mid i \in \mathbb{N})$ be the list of all proper quotients of G which are H -limit (Notice that the trivial group is a proper quotient of G). Then every G_i is generated by $f_i(\bar{a})$. As before, we let

$$P(\bar{x}) = \{w(\bar{x}) \mid w \text{ is a word such that } G \models w(\bar{a}) = 1\},$$

$$P_i(\bar{x}) = \{w(\bar{x}) \mid w \text{ is a word such that } G_i \models w(f_i(\bar{a})) = 1\}.$$

Since H is equationally noetherian, there exist a finite subsets $S(\bar{x}) \subseteq P(\bar{x})$, $S_i(\bar{x}) \subseteq P_i(\bar{x})$ such that

$$H \models \forall \bar{x} (S(\bar{x}) = 1 \Rightarrow w(\bar{x}) = 1), \text{ for any } w \in P(\bar{x}),$$

$$H \models \forall \bar{x} (S_i(\bar{x}) = 1 \Rightarrow w(\bar{x}) = 1), \text{ for any } w \in P_i(\bar{x}).$$

By Theorem 5.5, for any G_i , there exists a quantifier-free formula $\phi_i(\bar{x})$ such that if A is an H -limit group, if $Rk(A) \geq Rk(G_i)$ and if \bar{d} is an n -generating tuple of A such that $A \models \phi_i(\bar{d})$, then $(G_i, f_i(\bar{a})) \cong (A, \bar{d})$.

Similarly, for G and \bar{a} , there exists a quantifier-free formula $\phi(\bar{x})$ such that if A is an H -limit group such that $Rk(A) \geq Rk(G)$ and if \bar{d} is an n -generating tuple of A such that $A \models \phi(\bar{d})$, then $(G, \bar{a}) \cong (A, \bar{d})$.

Let

$$\Gamma(\bar{c}) = Th_{\forall}(H) \cup \{\neg\phi_i(\bar{c}) \mid Rk(G_i) < Rk(G)\} \cup \{\neg\phi(\bar{c})\} \cup \{S(\bar{c}) = 1\}.$$

If $\Gamma(\bar{c})$ is not consistent, then

$$Th_{\forall}(H) \cup \{\neg\phi_i(\bar{c}) \mid Rk(G_i) < Rk(G)\} \cup \{S(\bar{c}) = 1\} \vdash \phi(\bar{c}),$$

and thus, if $f : G \rightarrow L$ is an H -limit quotient with $Rk(L) \geq \gamma$, then $L \models \phi(f(\bar{a}))$, and thus f is an embedding. Therefore by taking $(f : G \rightarrow 1)$ to be our sequence, we get the result.

So we suppose that $\Gamma(\bar{c})$ is consistent. We claim that

$$(1) \quad \Gamma(\bar{c}) \vdash \bigvee_{i \in I} S_i(\bar{c}) = 1, \text{ where } I = \{i \in \omega \mid Rk(G_i) \geq Rk(G)\}.$$

Let (\mathcal{M}, \bar{d}) be a model of $\Gamma(\bar{c})$ in the language $L(\bar{c})$. Let A be the subgroup of \mathcal{M} , generated by \bar{d} . Then A is an H -limit group and since $A \models S(\bar{d}) = 1$, there exists a morphism f from G to A which sends \bar{a} to \bar{d} . Also since $A \models \neg\phi(\bar{d})$, A is a proper quotient of G . Furthermore, $Rk(A) \geq Rk(G)$; because if $Rk(A) < Rk(G)$ then $A = G_i$ and $A \models \phi_i(\bar{d})$ for some i , a contradiction. This ends the proof of our claim.

By compactness and (1), we get

$$\Gamma(\bar{c}) \vdash S_{i_1}(\bar{c}) = 1 \vee \dots \vee S_{i_m}(\bar{c}) = 1.$$

Let $L_j = G_{i_j}$ for $1 \leq j \leq m$ and $(f_i : G \rightarrow L_i \mid 1 \leq i \leq m)$ defined obviously. Then this sequence satisfy the desired conclusion. \square

The following theorem is a generalization of [OH05, Theorem 2.6]

Theorem 6.3. *Let H be an equationally noetherian group and G be a finitely generated H -limit group. Then for any finite subset $X \subseteq G \setminus \{1\}$ and for any ordinal $1 \leq \gamma \leq Rk(G)$, there exists an epimorphism $f : G \rightarrow L$ such that L is an H -limit group with $Rk(L) = \gamma$ and $1 \notin f(X)$.*

Before the proof we need some notions and results from [OH05].

Definition 6.4. A finitely generated H -limit group G is said *H -determined* if there exists a finite subset $X \subseteq G \setminus \{1\}$ such that for any morphism $f : G \rightarrow L$, where L is an H -limit group, if $1 \notin f(X)$ then f is an embedding.

Lemma 6.5. *Let H be an equationally noetherian group. A finitely generated H -limit group G is H -determined if and only if $Rk(G) = 1$.*

Proof. Let G be a finitely generated H -limit group which is also H -determined and let us prove that $Rk(G) = 1$. Write $G = \langle \bar{a} | P(\bar{a}) \rangle$. Then there exists a finite subset $S(\bar{x}) \subseteq P(\bar{x})$ such that $H \models \forall \bar{x} (S(\bar{x}) = 1 \Rightarrow w(\bar{x}) = 1)$, for any $w \in P(\bar{a})$. Let X given by the words $v_1(\bar{x}), \dots, v_m(\bar{x})$. We claim that the formula

$$\phi(\bar{c}) \equiv (S(\bar{c}) = 1 \wedge \bigwedge_{1 \leq i \leq m} v_i(\bar{x}) \neq 1),$$

supports $p_{\bar{a}, \forall}(G)$ over $P_{n,1}$.

Let L be an H -limit group, generated by \bar{b} , such that $L \models \phi(\bar{b})$. Then there exists a morphism $f : G \rightarrow L$ which sends \bar{a} to \bar{b} and $f(v_i(\bar{a})) \neq 1$. Since G is H -determined we find that f is an isomorphism and thus $(G, \bar{a}) \cong (L, \bar{b})$. Therefore $\phi(\bar{c})$ supports $p_{\bar{a}, \forall}(G)$ over $P_{n,1}$ and thus $Rk(G) = 1$.

Now suppose that $Rk(G) = 1$ and let us prove that G is H -determined. By theorem 5.5, there exists a quantifier-free formula $\phi(\bar{x})$ such that if L is an H -limit group, generated by \bar{b} , then $(G, \bar{a}) \cong (L, \bar{b})$ if and only if $L \models \phi(\bar{b})$ and $Rk(L) \geq Rk(G) = 1$. Clearly, replacing ϕ by a primitive quantifier-free formula, one can assume that ϕ is primitive quantifier-free. Then

$$\phi(\bar{x}) \equiv (\bigwedge_{w \in W} w(\bar{x}) = 1 \wedge \bigwedge_{v \in V} v(\bar{x}) \neq 1),$$

where W, V are finite sets of words.

Therefore, if $f : G \rightarrow L$ is a morphism, with L is H -limit and $f(v(\bar{a})) \neq 1$, for any $v \in V$, then f is an embedding. Thus G is H -determined as desired. \square

We will also need the following.

Theorem 6.6. [OH05] *Let H be an equationally noetherian group and G a non-trivial finitely generated H -limit group. Then for any finite subset $X \subseteq G \setminus \{1\}$ there exists an epimorphism $f : G \rightarrow L$ where L is an H -determined group such that $1 \notin f(X)$.* \square

Proof of Theorem 6.3.

If $\gamma = 1$, then the theorem is a consequence of Lemma 6.5 and Theorem 6.6. So we suppose that $\gamma \geq 2$.

Suppose that G is n -generated. Let $((G_i, \bar{a}_i) \mid \bar{a}_i \text{ generates } G_i, |\bar{a}_i| = n, i \in \omega)$ be the list of all H -limit groups, up to isomorphism, such that $1 \leq Rk(G_i) < \gamma$.

By theorem 5.5, for any (G_i, \bar{a}_i) , there exists a quantifier-free formula $\phi_i(\bar{x})$ such that if A is an H -limit group, if $Rk(A) \geq Rk(G_i)$ and if \bar{d} is an n -generating tuple of A such that $A \models \phi_i(\bar{d})$, then $(G_i, \bar{a}_i) \cong (A, \bar{d})$.

Let

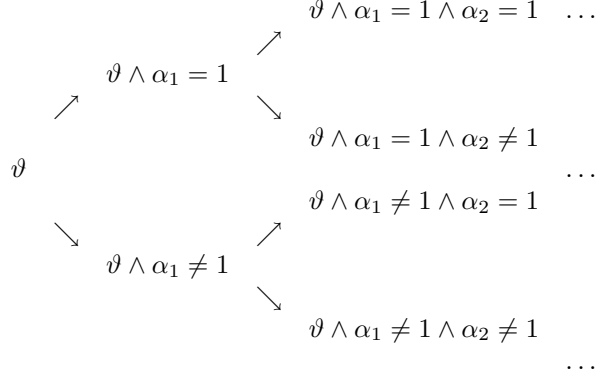
$$\Gamma(\bar{c}) = Th_{\forall}(H) \cup \{\neg \phi_i(\bar{c}) \mid i \in \omega\}.$$

Then $\Gamma(\bar{c})$ is consistent as (G, \bar{a}) is model of $\Gamma(\bar{c})$, for any generating n -tuple \bar{a} of G . Now we prove the following claim.

Claim. *For any primitive-quantifier-free formula $\vartheta(\bar{x})$ such that $\Gamma(\bar{c}) \cup \{(\vartheta(\bar{c}))\}$ is consistent, there exists a primitive-quantifier-free formula $\xi(\bar{x})$ such that $\Gamma(\bar{c}) \cup \{(\vartheta(\bar{c}) \wedge \xi(\bar{c}))\}$ is consistent and for any word $w(\bar{x})$ on the variables $\bar{x} = \{x_1, \dots, x_n\}$ and their inverses one has*

$$\Gamma(\bar{c}) \vdash (\vartheta(\bar{c}) \wedge \xi(\bar{c}) \Rightarrow w(\bar{c}) = 1) \text{ or } \Gamma(\bar{c}) \vdash (\vartheta(\bar{c}) \wedge \xi(\bar{c}) \Rightarrow w(\bar{c}) \neq 1).$$

Proof. Let $\vartheta(\bar{x})$ be a primitive-quantifier-free formula such that $\Gamma(\bar{c}) \cup \{(\vartheta(\bar{c}))\}$ is consistent and suppose towards a contradiction that $\vartheta(\bar{x})$ does not satisfies the conclusions of the claim. We are going to construct a tree. By hypothesis there exists a word $\alpha_1(\bar{x})$ such that $\Gamma(\bar{c}) \cup \{(\vartheta(\bar{c}) \wedge \alpha_1(\bar{c}) = 1)\}$ and $\Gamma(\bar{c}) \cup \{(\vartheta(\bar{x}) \wedge \alpha_1(\bar{c}) \neq 1)\}$ are consistent (to simplify notation we omit \bar{c}). We can do the same thing with $\vartheta \wedge \alpha_1 = 1$ and $\vartheta \wedge \alpha_1 \neq 1$. Thus we have:



Therefore, by compactness, every branch in the tree is consistent with $\Gamma(\bar{c})$. Since there exists 2^{\aleph_0} branch we get 2^{\aleph_0} nonisomorphic finitely generated models of $\Gamma(\bar{c})$ and thus $\text{Th}_{\forall}(H)$ has 2^{\aleph_0} nonisomorphic finitely generated models. A contradiction. \square

Write $G = \langle \bar{a} | P(\bar{a}) \rangle$, $|\bar{a}| = n$, and let $S(\bar{a}) \subseteq P(\bar{a})$ be a finite set such that

$$H \models \forall \bar{x} (S(\bar{x}) = 1 \Rightarrow w(\bar{x}) = 1), \text{ for any } w \in P(\bar{x}).$$

Let $X \subseteq G$ be a finite subset, given by words $v_1(\bar{x}), \dots, v_n(\bar{x})$ such that $G \models \bigwedge_{1 \leq i \leq n} v_i(\bar{a}) \neq 1$. Let $\vartheta(\bar{x}) = (S(\bar{x}) = 1 \wedge \bigwedge_{1 \leq i \leq n} v_i(\bar{x}) \neq 1)$.

Since $(G, \bar{a}) \models \vartheta(\bar{a})$, $\Gamma(\bar{c}) \cup \{\vartheta(\bar{c})\}$ is consistent. Therefore, by the claim above, there exists a primitive-quantifier-free formula $\xi(\bar{x})$ such that $\Gamma(\bar{c}) \cup \{(\vartheta(\bar{c}) \wedge \xi(\bar{c}))\}$ is consistent and for any word $w(\bar{x})$ on the variables $\bar{x} = \{x_1, \dots, x_n\}$ and their inverses one has

$$(*) \quad \Gamma(\bar{c}) \vdash (\vartheta(\bar{c}) \wedge \xi(\bar{c}) \Rightarrow w(\bar{c}) = 1) \text{ or } \Gamma(\bar{c}) \vdash (\vartheta(\bar{c}) \wedge \xi(\bar{c}) \Rightarrow w(\bar{c}) \neq 1).$$

Let (\mathcal{M}, \bar{b}) a model of $\Gamma(\bar{c}) \cup \{(\vartheta(\bar{c}) \wedge \xi(\bar{c}))\}$, and let L be the subgroup of \mathcal{M} generated by \bar{b} . We claim that L satisfies the desired property. Clearly, L is an H -limit group. Since $L \models \vartheta(\bar{b})$, there exists an epimorphism $f : G \rightarrow L$ such that $f(\bar{a}) = \bar{b}$ and $1 \notin f(X)$.

Therefore, it remains to show that $Rk(L) = \gamma$. Since $L \models \neg \phi_i(\bar{b})$ we have $Rk(L) \geq \gamma$. By the property $(*)$, we see that the formula $\vartheta(\bar{x}) \wedge \xi(\bar{x})$ supports $p_{\bar{b}, \forall}(L)$ over $\mathcal{K}(\text{Th}_{\forall}(H) | Q_{n, \gamma})$ and thus by the definition of the rank we get $Rk(L) = \gamma$ as desired. \square

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